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LLNL-TR-637397

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May 24, 2013

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This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

On High Order Finite-Difference Metric Discretizations Satisfying GCL on Moving and Deforming Grids

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May 23, 2013

1 Introduction

The purpose of this short communication is to present a straightforward extension of the work of Vinokur & Yee for 3D curvilinear moving grids in the high order finite-difference frame work which include deforming grids that satisfy the GCL (Geometric Conservation Law). The main ingredient that was used in Vinokur & Yee is couched on the commutative property of mixed difference operators for metric evaluations. This property can be applied to mixed difference operators including time metric evaluations for deforming grids. The natural and obvious candidates to satisfy the commutative property are linear time and spatial difference operators. For separable difference schemes using the method of lines (MOL) approach, all first-order linear operators and high order linear multiple step methods (LMMs) temporal discretizations satisfy the GCL. Examples of spatial operators are all orders of central difference operators or any order of linear difference operators. For in-separable Lax-Wendroff-type difference schemes (second-order or higher) some analysis is needed to make sure the desired accuracy is maintain.

[*]Lawrence Livermore National Laboratory, Livermore, CA 94551. Work by the first author performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344. This is contribution LLNL-TRNL-xxxxxx.

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Recently, Abe et al. [1] made use of the commutative property presented in Vinokur & Yee to construct their conservative metric evaluation that satisfies the GCL identity. The first purpose of this note is to illustrate that the Abe et al. [1] formulation can be simplified to just half of the needed terms to satisfy the GCL identity. In addition, Abe et al. made use of the linear difference operators that automatically satisfy the commutative property for first order time discretization. For general multistage Runge-Kutta (RK) methods that are higher than first order, the majority are not explicitly linear. Special construction of higher than first-order RK methods is needed to insure the commutative property and maintain the desired order of accuracy. In all of the Abe et al. examples they used a standard second-order RK method without performing the systematic analysis to see if the commutative property is satisfied. In addition it is uncertain whether or not this RK method maintains second-order accuracy on their time metric evaluation. The second purpose of this note is to show that some of the typical explicit and implicit LMMs popularized by Beam & Warming [2] do satisfy the commutative property. A construction of two-stage RK methods with the commutative property and first-order and second-order accuracy is included. The following expands the discussion further.

We consider a finite difference approximation of partial differential equations, where a curvilinear grid discretizes the computational domain. The grid is generated by a coordinate mapping from a reference cube $(\xi, \eta, \zeta) \in [0,1]^3$ with time τ to the physical domain $(x, y, z) \in \Omega$ with time t, and is given by an invertible mapping

$$x = x(\xi, \eta, \zeta, \tau) \tag{1}$$
$$y = y(\xi, \eta, \zeta, \tau) \tag{2}$$
$$z = z(\xi, \eta, \zeta, \tau) \tag{3}$$
$$t = \tau. \tag{4}$$

The mapping transforms the conservation law

$$u_t + f(u)_x + g(u)_y + h(u)_z = 0$$

on Ω into

$$\begin{aligned}(Ju)_\tau + (J\xi_x f + J\xi_y g + J\xi_z h + J\xi_t u)_\xi + \\ (J\eta_x f + J\eta_y g + J\eta_z h + J\eta_t u)_\eta + (J\zeta_x f + J\zeta_y g + J\zeta_z h + J\zeta_t u)_\zeta = 0\end{aligned} \tag{5}$$

on the domain $[0,1]^3$. Partial derivatives are denoted by subscript notation. For example, ξ_x denotes the partial derivative $\partial\xi/\partial x$. The Jacobian of the mapping, J, is the determinant of the matrix of partial derivatives $\partial(x,y,z)/\partial(\xi,\eta,\zeta)$. The derivation of the conservative form (5) makes use of the so called geometric conservation laws

$$(J\xi_x)_\xi + (J\eta_x)_\eta + (J\zeta_x)_\zeta = 0 \tag{6}$$

$$(J\xi_y)_\xi + (J\eta_y)_\eta + (J\zeta_y)_\zeta = 0 \tag{7}$$

$$(J\xi_z)_\xi + (J\eta_z)_\eta + (J\zeta_z)_\zeta = 0 \tag{8}$$

$$J_\tau + (J\xi_t)_\xi + (J\eta_t)_\eta + (J\zeta_t)_\zeta = 0. \tag{9}$$

Here, the term 'conservation law' is somewhat misleading, since (6)–(9) are really identities that are satisfied by any given differentiable functions (1)–(3). It is straightforward to see that (6)–(9) guarantee that any constant u is a solution of (5). We will show how to discretize (5) in such a way that (6)–(9) are satisfied exactly by the discretization. When (6)–(9) are satisfied exactly, regions of constant u, such as a free stream state, will be preserved exactly. In general, a finite difference discretization only guarantees that a free stream state is constant up to the order of truncation error of the discretization. There are many examples where an approximation that has the exact free stream preserving property is advantageous.

Discretizations that satisfy the identities (6)–(8) were developed by [6] in non-symmetric form. Vinokur and Yee showed in [8] how a coordinate invariant, or symmetric, form can be devised. The work of Visbal and Gaitonde [9] and Ou and Jameson [4] is not symmetric and they do not make used of the commutative property on their metric construction. Recently, Abe et al. [1] developed a discretization that satisfies (9). Making use of the Vinokur and Yee symmetric formulation for moving grids in the high order finite-difference frame work, it will be shown that we can straightforwardly extend the idea of Vinokur and Yee to include deforming grid time metrics that satisfy the GCL identity. From our formulation we will show how the construction by Abe et al. can be considerably simplified with just half of their indicated terms. In addition, we will extend the results to multi-stage Runge-Kutta time discretizations.

2 Geometric conservation laws

We introduce the vector notation

$$\mathbf{S}^{(\xi)} = J \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix} \quad \mathbf{S}^{(\eta)} = J \begin{pmatrix} \eta_x \\ \eta_y \\ \eta_z \end{pmatrix} \quad \mathbf{S}^{(\zeta)} = J \begin{pmatrix} \zeta_x \\ \zeta_y \\ \zeta_z \end{pmatrix} \tag{10}$$

In this notation, (6)–(8) become

$$(\mathbf{S}^{(\xi)})_\xi + (\mathbf{S}^{(\eta)})_\eta + (\mathbf{S}^{(\zeta)})_\zeta = \mathbf{0}. \tag{11}$$

The vector $\mathbf{S}^{(\xi)}$ can be evaluated in terms of the derivatives $\mathbf{r}_\xi = (x_\xi \ y_\xi \ z_\xi)$, because of the relations

$$\mathbf{S}^{(\xi)} = \mathbf{r}_\eta \times \mathbf{r}_\zeta \qquad \mathbf{S}^{(\eta)} = \mathbf{r}_\zeta \times \mathbf{r}_\xi \qquad \mathbf{S}^{(\zeta)} = \mathbf{r}_\xi \times \mathbf{r}_\eta. \tag{12}$$

For a derivation of these formulas, see [7]. The coordinate invariant form by Vinokur and Yee is obtained by rewriting (12) as the mathematically equivalent

$$\mathbf{S}^{(\xi)} = \frac{1}{2}\left((\mathbf{r} \times \mathbf{r}_\zeta)_\eta - (\mathbf{r} \times \mathbf{r}_\eta)_\zeta\right) \tag{13}$$

$$\mathbf{S}^{(\eta)} = \frac{1}{2}\left((\mathbf{r} \times \mathbf{r}_\xi)_\zeta - (\mathbf{r} \times \mathbf{r}_\zeta)_\xi\right) \tag{14}$$

$$\mathbf{S}^{(\zeta)} = \frac{1}{2}\left((\mathbf{r} \times \mathbf{r}_\eta)_\xi - (\mathbf{r} \times \mathbf{r}_\xi)_\eta\right) \tag{15}$$

before approximating. Let the derivatives with respect to $(\xi \ \eta \ \zeta)$ in (13)–(15) be approximated by difference operators $D^{(\xi)}$, $D^{(\eta)}$, and $D^{(\zeta)}$ respectively. For example, the approximation of (13) is

$$\mathbf{S}_h^{(\xi)} = \frac{1}{2}\left(D^{(\eta)}(\mathbf{r} \times D^{(\zeta)}\mathbf{r}) - D^{(\zeta)}(\mathbf{r} \times D^{(\eta)}\mathbf{r})\right). \tag{16}$$

The approximations of (14) and (15) are similar. We use the subscript h to denote a quantity that has been discretized on a grid. It is then straightforward that if the operators along the different coordinate directions commute, e.g., $D^{(\xi)}D^{(\eta)} = D^{(\eta)}D^{(\xi)}$, then the finite difference approximation satisfies the discretized (11),

$$D^{(\xi)}\mathbf{S}_h^{(\xi)} + D^{(\eta)}\mathbf{S}_h^{(\eta)} + D^{(\zeta)}\mathbf{S}_h^{(\zeta)} = \mathbf{0} \tag{17}$$

exactly. The commutative property holds for standard centered difference operators; see e.g., [8] for a proof. Note that here the difference operators used in (16) and (17) are assumed to be the same. However, a sufficient condition to satisfy (17) is that only the 'outer' difference operators in (16) are the same as used in (17). The 'inner' operators in (16) could be replaced by some other operators $\widetilde{D}^{(\xi)}$, $\widetilde{D}^{(\eta)}$, and $\widetilde{D}^{(\zeta)}$, so that the metric discretization (16) instead becomes

$$\mathbf{S}_h^{(\xi)} = \frac{1}{2}\left(D^{(\eta)}(\mathbf{r} \times \widetilde{D}^{(\zeta)}\mathbf{r}) - D^{(\zeta)}(\mathbf{r} \times \widetilde{D}^{(\eta)}\mathbf{r})\right). \tag{18}$$

This possibility is discussed by Deng et al. in [3], who conclude that the accuracy is better when the same difference operators are used everywhere. We remark here that for spatial discretizations, it is a standard procedure to use the same difference metric operator for all spatial directions, even if different spatial discretizations in each of the inviscid flux derivative spatial directions might be employed. Here, we only consider separable finite difference schemes of the MOL type. Temporal finite difference operator construction is different from the spatial discretizations. Often, different orders and different types of finite difference temporal discretizations are employed from the spatial difference operators. This MOL discretizations are employed in our high order metric constructions as well.

Next, we show a natural generalization of the above to obtain a metric discretization that satisfies (9) exactly. In vector notation it holds that,

$$J = \mathbf{r}_\xi \cdot (\mathbf{r}_\eta \times \mathbf{r}_\zeta),$$

and

$$J\xi_t = -\mathbf{r}_\tau \cdot (\mathbf{r}_\eta \times \mathbf{r}_\zeta) \tag{19}$$

$$J\eta_t = -\mathbf{r}_\tau \cdot (\mathbf{r}_\zeta \times \mathbf{r}_\xi) \tag{20}$$

$$J\zeta_t = -\mathbf{r}_\tau \cdot (\mathbf{r}_\xi \times \mathbf{r}_\eta) \tag{21}$$

The terms in the identity (9) can be rewritten in a way similar to (13)–(15) by noting that

$$J = \mathbf{r}_\xi \cdot (\mathbf{r}_\eta \times \mathbf{r}_\zeta) = \frac{1}{3}\left((\mathbf{r} \cdot (\mathbf{r}_\eta \times \mathbf{r}_\zeta))_\xi + (\mathbf{r} \cdot (\mathbf{r}_\zeta \times \mathbf{r}_\xi))_\eta + (\mathbf{r} \cdot (\mathbf{r}_\xi \times \mathbf{r}_\eta))_\zeta\right). \tag{22}$$

Applying (22) to each of (19)–(21) with obvious modifications gives

$$J\xi_t = -\frac{1}{3}\left((\mathbf{r}\cdot(\mathbf{r}_\eta \times \mathbf{r}_\zeta))_\tau + (\mathbf{r}\cdot(\mathbf{r}_\zeta \times \mathbf{r}_\tau))_\eta + (\mathbf{r}\cdot(\mathbf{r}_\tau \times \mathbf{r}_\eta))_\zeta\right) \tag{23}$$

$$J\eta_t = -\frac{1}{3}\left((\mathbf{r}\cdot(\mathbf{r}_\zeta \times \mathbf{r}_\xi))_\tau + (\mathbf{r}\cdot(\mathbf{r}_\xi \times \mathbf{r}_\tau))_\zeta + (\mathbf{r}\cdot(\mathbf{r}_\tau \times \mathbf{r}_\zeta))_\xi\right) \tag{24}$$

$$J\zeta_t = -\frac{1}{3}\left((\mathbf{r}\cdot(\mathbf{r}_\xi \times \mathbf{r}_\eta))_\tau + (\mathbf{r}\cdot(\mathbf{r}_\eta \times \mathbf{r}_\tau))_\xi + (\mathbf{r}\cdot(\mathbf{r}_\tau \times \mathbf{r}_\xi))_\eta\right). \tag{25}$$

Let the discretization of J, $J\xi_t$, $J\eta_t$, and $J\zeta_t$ be done by replacing all derivatives with respect to $(\xi\ \eta\ \zeta\ \tau)$ in (22) and (23)–(25) by commuting finite difference operators, $D^{(\xi)}$, $D^{(\eta)}$, $D^{(\zeta)}$, and $D^{(\tau)}$, e.g., (23) would be discretized by

$$\begin{aligned}(J\xi_t)_h = -\frac{1}{3}\big(&D^{(\tau)}(\mathbf{r}\cdot(D^{(\eta)}\mathbf{r}\times D^{(\zeta)}\mathbf{r}))+\\ &D^{(\eta)}(\mathbf{r}\cdot(D^{(\zeta)}\mathbf{r}\times D^{(t)}\mathbf{r})) + D^{(\zeta)}(\mathbf{r}\cdot(D^{(t)}\mathbf{r}\times D^{(\eta)}\mathbf{r}))\big).\end{aligned} \tag{26}$$

It is then straightforward to verify that this discretization satisfies the following discrete version of (9),

$$D^{(\tau)}J_h + D^{(\xi)}(J\xi_t)_h + D^{(\eta)}(J\eta_t)_h + D^{(\zeta)}(J\zeta_t)_h = 0. \tag{27}$$

The metric discretization given in [1] also satisfies (27). However, in [1], equation (22) is further decomposed by applying (13)–(15) to the vector products. This leads to the Jacobian being rewritten as

$$\begin{aligned}J = \mathbf{r}_\xi\cdot(\mathbf{r}_\eta\times\mathbf{r}_\zeta) = \frac{1}{3}\left((\mathbf{r}\cdot(\mathbf{r}_\eta\times\mathbf{r}_\zeta))_\xi + (\mathbf{r}\cdot(\mathbf{r}_\zeta\times\mathbf{r}_\xi))_\eta + (\mathbf{r}\cdot(\mathbf{r}_\xi\times\mathbf{r}_\eta))_\zeta\right) = \\ \frac{1}{3}\Bigg((\mathbf{r}\cdot(\frac{1}{2}((\mathbf{r}\times\mathbf{r}_\zeta)_\eta - (\mathbf{r}\times\mathbf{r}_\eta)_\zeta)))_\xi + (\mathbf{r}\cdot(\frac{1}{2}((\mathbf{r}\times\mathbf{r}_\xi)_\zeta - (\mathbf{r}\times\mathbf{r}_\zeta)_\xi)))_\eta + \\ (\mathbf{r}\cdot(\frac{1}{2}((\mathbf{r}\times\mathbf{r}_\eta)_\xi - (\mathbf{r}\times\mathbf{r}_\xi)_\eta)))_\zeta\Bigg)\end{aligned} \tag{28}$$

before discretizing.Formulas (23)–(25) are rewritten in a similar way. Hence, this discretization requires double the number of terms.

3 Time integration

More moving deformable grids, it is important to distinguish between the two cases: (a) Grid mapping is a known, given, function of time, and (b) The grid depends on the computed solution, and therefore needs to be solved for in time together with the solution of the PDE. Here our discussion focuses on (a). We are not sure if any of the theory could be used for (b) as well.

The forward time difference

$$D^{(\tau)}u^n = (u^{n+1} - u^n)/\Delta t \tag{29}$$

is a linear difference operator, and hence it can be used in time to define a discretization that satisfies (27), as outlined in the previous section. Here, superscript n denotes time level. We assume that time has been discretized with a uniform time step Δt, so that the time levels are $t_n = n\Delta t$, $n = 0, 1, \ldots$. Similarly, time discretizations that can be written of the form

$$Dy_n = f(y^*, t_*),$$

for some y^*, t_*, when applied to the ODE $y_t = f(y, t)$, allow straightforward inclusion of the moving grid terms. The operator D is any finite difference operator approximating the time derivative. Examples of methods of this form are the backward Euler, leap-frog, and backward differentiation (BDF) methods. Furthermore, it is possible to include the moving grid terms in general LMMs to obtain the exact conservation (27) in the sense that the local truncation error is exactly zero for constant u. For higher order methods it is important to verify that the order of accuracy is not degraded by adding the moving grid terms.

For higher order of accuracy Runge-Kutta (RK) methods are standard choices. RK methods are not explicitly linear difference operators, but by making use of (27) it is possible to choose the metric at the different RK stages to perfectly preserve a free stream state on a moving and deforming grid. For simplicity we assume that $f(u) = g(u) = h(u) = 0$, so that (5) becomes

$$(Ju)_\tau + (J\xi_t u)_\xi + (J\eta_t u)_\eta + (J\zeta_t u)_\zeta = 0. \tag{30}$$

Denote

$$s_h(\mathbf{r}(\tau_2), \mathbf{r}(\tau_1), u) = D^{(\xi)}((J\xi_t)_h u) + D^{(\eta)}((J\eta_t)_h u) + D^{(\zeta)}((J\zeta_t)_h u),$$

where the time differences of the metric are approximated by differences between $\mathbf{r}(\tau_2)$ and $\mathbf{r}(\tau_1)$, and where grid terms not differentiated in time are evaluated at τ_1. For example,

$$\begin{aligned}(J\xi_t)_h = -\frac{1}{3\Delta t}\big(&\mathbf{r}(\tau_2)\cdot(D^{(\eta)}\mathbf{r}(\tau_2)\times D^{(\zeta)}\mathbf{r}(\tau_2)) - \\ &\mathbf{r}(\tau_1)\cdot(D^{(\eta)}\mathbf{r}(\tau_1)\times D^{(\zeta)}\mathbf{r}(\tau_1)) + \\ &D^{(\eta)}(\mathbf{r}(\tau_1)\cdot(D^{(\zeta)}\mathbf{r}(\tau_1)\times(\mathbf{r}(\tau_2)-\mathbf{r}(\tau_1)))) + \\ &D^{(\zeta)}(\mathbf{r}(\tau_1)\cdot((\mathbf{r}(\tau_2)-\mathbf{r}(\tau_1))\times D^{(\eta)}\mathbf{r}(\tau_1)))\big)\,. \end{aligned} \quad (31)$$

The grid $\mathbf{r}(\tau)$ can be evaluated at any τ, because it is assumed to be a given function. Note the simplified notation $\mathbf{r}(\tau) = \mathbf{r}(\tau,\xi,\eta,\zeta)$, suppressing the spatial arguments of the grid mapping. When u is a constant, it follows from (27) that

$$\Delta t s_h(\mathbf{r}(\tau_2),\mathbf{r}(\tau_1),u) = (-J_h(\tau_2)+J_h(\tau_1))u, \quad (32)$$

where the Jacobian can be evaluated at any τ, since it is a function of the grid. The spatial derivatives in (31) are evaluated at τ_1. Exchanging τ_1 and τ_2 gives an approximation with the spatial part centered at τ_2 that satisfies

$$-\Delta t s_h(\mathbf{r}(\tau_1),\mathbf{r}(\tau_2),u) = (-J_h(\tau_2)+J_h(\tau_1))u. \quad (33)$$

The two-stage RK method on conservation form

$$J_h(\tau_{n+1})u^{(1)} = J_h(\tau_n)u^n - \Delta t s_h(\mathbf{r}(\tau_{n+1}),\mathbf{r}(\tau_n),u^n) \quad (34)$$

$$(2J_h(\tau_{n+1}) - J_h(\tau_n))u^{(2)} = J_h(\tau_{n+1})u^{(1)} + \Delta t s_h(\mathbf{r}(\tau_n),\mathbf{r}(\tau_{n+1}),u^{(1)}) \quad (35)$$

$$J_h(\tau_{n+1})u^{n+1} = \frac{1}{2}\left(J_h(\tau_n)u^n + (2J_h(\tau_{n+1}) - J_h(\tau_n))u^{(2)}\right) \quad (36)$$

is second order accurate and preserves constants perfectly. When the metric is stationary, (34)–(36) is the standard second order TVD RK method [5].

To see that constants are left unchanged, assume that u^n is given and constant. Equation (32) applied to the first stage gives that $u^{(1)} = u^n$. Similarly, we obtain for the second stage $u^{(2)} = u^{(1)}$, and hence $u^{(2)} = u^n$, so that by (36) $u^{n+1} = u^n$. Second order accuracy is shown in Appendix A.

Note that there are many different ways that the moving metric could be discretized in a RK method. For example, the method

$$J_h(\tau_{n+1})u^{(1)} = J_h(\tau_n)u^n - \Delta t s_h(\mathbf{r}(\tau_{n+1}),\mathbf{r}(\tau_n),u^n) \quad (37)$$

$$J_h(\tau_{n+2})u^{(2)} = J_h(\tau_{n+1})u^{(1)} - \Delta t s_h(\mathbf{r}(\tau_{n+2}),\mathbf{r}(\tau_{n+1}),u^{(1)}) \quad (38)$$

$$(J_h(\tau_{n+2}) + J_h(\tau_n))u^{n+1} = J_h(\tau_n)u^n + J_h(\tau_{n+2})u^{(2)} \quad (39)$$

is another generalization of the standard second order TVD RK method that preserves constants, but (37)–(39) is only first order accurate.

In summary, a systematic formulation of conservative symmetric finite-difference metric discretizations that satisfy the GCL identity exactly in moving deformable grids is presented. A wide class of temporal metric discretizations that satisfy the GCL identity is discussed. In general, higher than first-order RK methods are not explicitly linear difference operators. Construction of two multistage RK method that satisfy the GCL is included. Extensive numerical experiments on practical test cases are planned.

A Second order of accuracy

To demonstrate the accuracy of (34)–(36), consider the scheme applied to the equation $(Ju)_t + D(a_t u) = 0$, where $J = J(x,t)$ and $a = a(x,t)$ are known functions, and $u = u(x,t)$ is the unknown. D is an unspecified difference operator acting only in the x-direction. The truncation error of (34)–(36), denoted τ, is obtained by inserting the exact solution, u, into the RK method,

$$\begin{aligned}\tau = J_h^{n+1}u^{n+1} - \frac{1}{2}(J_h^n u^n + J_h^{n+1}u^{(1)} - D((a^{n+1}-a^n)u^{(1)})) = \\ J_h^{n+1}u^{n+1} - \frac{1}{2}(J_h^n u^n + J_h^n u^n - D((a^{n+1}-a^n)u^n) - \\ D(\frac{a^{n+1}-a^n}{J_h^{n+1}}(J_h^n u^n - D((a^{n+1}-a^n)u^n)))) \quad (40)\end{aligned}$$

and performing a Taylor expansion in the time variable around t_n. The spatial accuracy is assumed to be order two or higher. The part of the truncation error from the operator D will not be considered below. To simplify the notation, denote $J = J_h^n$, $u = u^n$, and $a = a^n$. The time step is denoted by h.

Taylor expansion up to second order gives

$$\tau = (J + hJ_t + \frac{h^2}{2}J_{tt})(u + hu_t + \frac{h^2}{2}u_{tt}) - Ju - \frac{1}{2}(D((ha_t + \frac{h^2}{2}a_{tt})u) -$$
$$D((ha_t + \frac{h^2}{2}a_{tt})(\frac{1}{J} - h\frac{J_t}{J^2})(Ju - D(ha_t u)))) + \mathcal{O}(h^3) =$$
$$h(Ju)_t + \frac{h}{2}D(a_t u) + \frac{h^2}{2}(Ju)_{tt} + \frac{h^2}{4}D(a_{tt}u) + T + \mathcal{O}(h^3) \quad (41)$$

where

$$T = \frac{1}{2}D\left((ha_t + \frac{h^2}{2}a_{tt})(u - h\frac{1}{J}D(a_t u) - h\frac{J_t}{J}u)\right) =$$
$$\frac{1}{2}D\left(ha_t u - h^2 a_t \frac{1}{J}D(a_t u) - h^2 a_t \frac{J_t}{J}u + \frac{h^2}{2}a_{tt}u\right) + \mathcal{O}(h^3). \quad (42)$$

Substitute $D(a_t u) = -(Ju)_t$ in the second term to obtain

$$T = \frac{1}{2}D\left(ha_t u + h^2 a_t \frac{1}{J}(Ju)_t - h^2 a_t \frac{J_t}{J}u + \frac{h^2}{2}a_{tt}u\right) + \mathcal{O}(h^3) =$$
$$\frac{1}{2}D\left(ha_t u + h^2 a_t u_t + \frac{h^2}{2}a_{tt}u\right) + \mathcal{O}(h^3) \quad (43)$$

The final result becomes

$$\tau = h(Ju)_t + \frac{h}{2}D(a_t u) + \frac{h^2}{2}(Ju)_{tt} + \frac{h^2}{4}D(a_{tt}u) + \frac{h}{2}D(a_t u) + \frac{h^2}{2}D(a_t u_t) +$$
$$\frac{h^2}{4}D(a_{tt}u) + \mathcal{O}(h^3) \quad (44)$$

Collecting terms and making use of the equation and its derivative, $(Ju)_{tt} + D((a_t u)_t) = 0$, yield the final result

$$\tau = \mathcal{O}(h^3)$$

and, hence, the order of accuracy is two.

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